Convex Elicitation of Continuous Properties

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Empirical Risk Minimization (ERM)

Find hypothesis h^* so that

$$h^* = \arg\min_{h\in\mathcal{H}} \sum_{(x,y)\in\text{data}} L(h(x), y) \tag{1}$$

Definition 1: Elicits

A loss function $L : \mathcal{R} \times \mathcal{Y} \to \mathbb{R}$ *elicits* a property Γ if for all $p \in \mathcal{P}$,

$$\{\Gamma(p)\} = \arg\min_r \mathbb{E}_{Y\sim p} L(r, Y) \ .$$

Definition 2: Identifies

$V : \mathcal{R} \times \mathcal{Y} \to \mathbb{R}$ *identifies* $\Gamma : \mathcal{P} \to \mathcal{R}$ if, for all $r \in \mathring{\mathcal{R}}$ and $p \in \mathcal{P}$

$$\mathbb{E}_{Y\sim p}[V(r, Y)] = 0 \iff r = \Gamma(p)$$

Steinwart et al. (2014) and Lambert (2018) show identifiable $\iff$ elicitable for continuous, nowhere-locally-constant, real-valued properties.

Results

Theorem 1: Elicitable $\iff$ Convex elicitable

For $\mathcal{P} = \Delta(\mathcal{Y})$, let $\Gamma : \mathcal{P} \to \mathcal{R}$ be a continuous, nowhere-locally-constant property which is identified by a bounded and oriented $V : \mathcal{R} \times \mathcal{Y} \to \mathbb{R}$. If $\mathcal{F} = \{V(\cdot, y)\}_{y\in\mathcal{Y}}$ satisfies Condition 1 (below), then Γ is convex elicitable.

Proof Intuition

- $L(r, y) = \int_0^r \lambda(x)V(x, y)dx$ elicits Γ for $\lambda : \mathcal{R} \to \mathbb{R}_{>0}$
  - Steinwart et al. (2014) and Lambert (2018)
- If $\lambda(r)V(r, y)$ is increasing in $\mathcal{R}$ for all $y \in \mathcal{Y}$, then $\mathbb{E}_p L(r, Y)$ is a convex combination of convex functions.
- *What are the conditions on $\{V(r, y)\}_{y\in\mathcal{Y}}$ so that we can design λ where the above is true?*

Consider the following three simple Conditions:

Condition 1'.[1] Every $f \in \mathcal{F}$ is continuously differentiable.
Condition 2'. Each $f \in \mathcal{F}$ has a single zero, and moves from negative to positive.
Condition 3'. When $f > 0 > g$, the ratio g/f is increasing.

Two outcomes

- Consider $f > 0 > g$

$$\lambda(r) = (-f(r)g(r))^{-1/2}.$$

Then

$$(\lambda f)(r) = \sqrt{-f(r)/g(r)} \qquad (\lambda g)(r) = \sqrt{-g(r)/f(r)}$$

- Focus on the "most decreasing" and "least increasing" functions.

Examples

Beta families (Buja et al. (2005))

$$V(r, y) = r^{\alpha-1}(1-r)^{\beta-1}(r-y),\ L(r, y) = \int_0^r z^{\alpha-1}(1-z)^{\beta-1}(z-y)dz$$

- Log loss ($\alpha = \beta = 0$) and squared loss ($\alpha = \beta = 1$).
- $\lambda(r) = r^{1/2-\alpha}(1-r)^{1/2-\beta}$ yields:

$$V'(r, y) = r^{1/2}(1-r)^{1/2}(r-y)$$
$$L'(r, y) = \arcsin(\sqrt{|y-r|}) - \sqrt{r(1-r)}$$

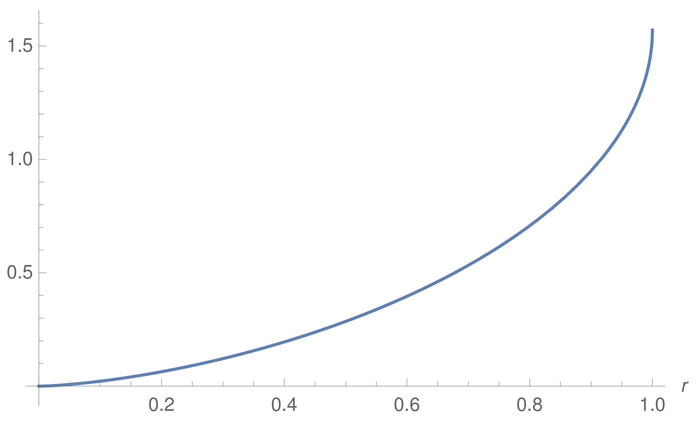

Figure 1: $L'(r, 0)$

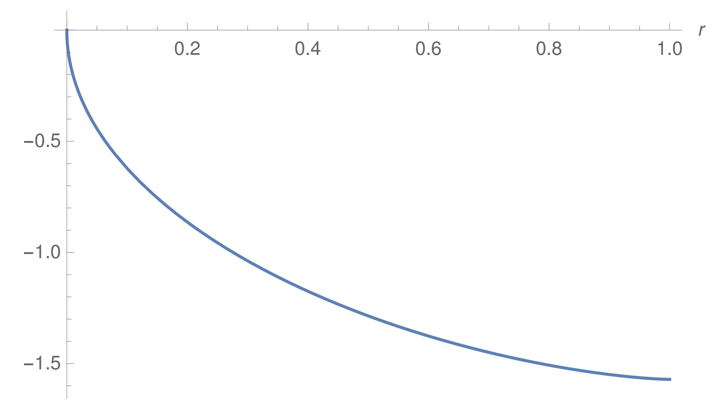

Figure 2: $L'(r, 1)$

A quadratic property

- Natural choice for $V(r, y)$ is $V(r, 1) = r - 1$, $V(r, 2) = \frac{1}{2} + r - r^2$, $V(r, 3) = r$.

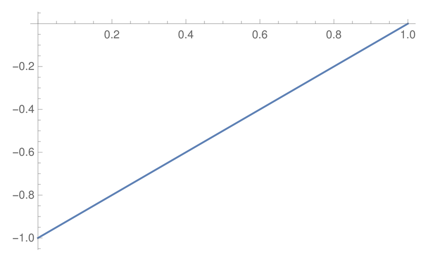

Figure 3: $V(\cdot, 1)$

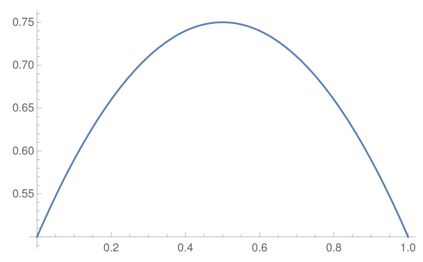

Figure 4: $V(\cdot, 2)$

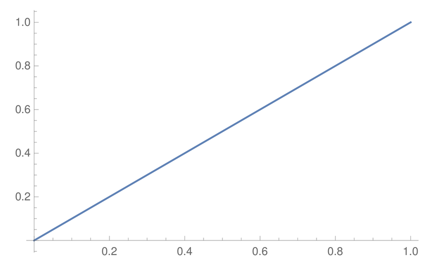

Figure 5: $V(\cdot, 3)$

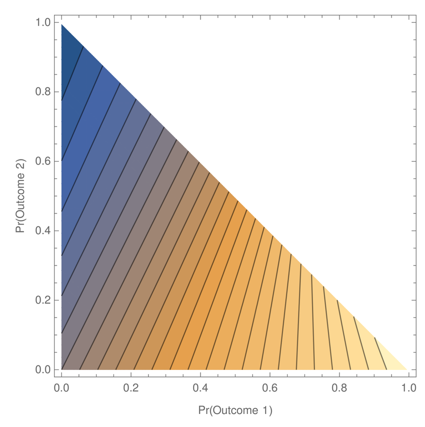

Figure 6: Level sets

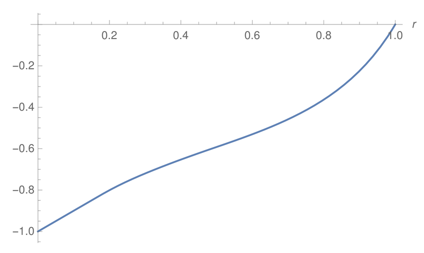

Figure 7: $\lambda(\cdot)V(\cdot, 1)$

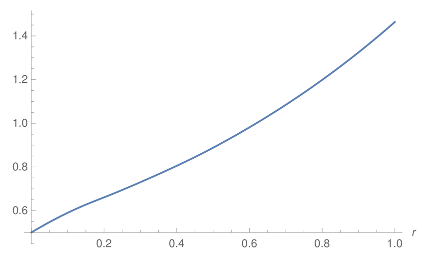

Figure 8: $\lambda(\cdot)V(\cdot, 2)$

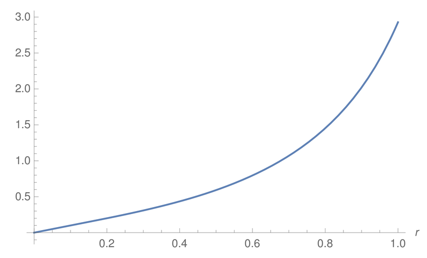

Figure 9: $\lambda(\cdot)V(\cdot, 3)$

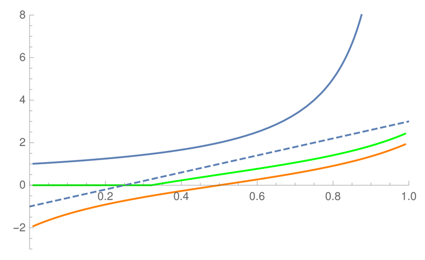

Figure 10: $\overline{m}, h, \underline{m}$

Constructing λ

- $L(r, y) = \int_0^r \lambda(x)V(x, y)dx$ elicits Γ given weight function $\lambda : \mathcal{R} \to \mathbb{R}_{>0}$.
- Design $\lambda(r)$ so that $\lambda(r)V(r, y)$ increasing in r for each $y \in \mathcal{Y}$
  - Find bounds on h.
  - Search over "simple" class of functions and select h that fits bounds

$$\lambda(r) = \exp\left(\int_0^r h(r)dr\right)$$

Scoring Rule Markets

- Lambert et al. (2008) generalizes the prediction market framework to *Scoring Rule Market* (SRM).
- Given loss $L(r, y)$ and initial central prediction r_0, each trader updates the central prediction $r_{t-1} \to r_t$, and suffers loss $L(r_t, y) - L(r_{t-1}, y)$.
- Abernethy and Frongillo (2011) and Frongillo and Waggoner (2018)
  - Tractable Trade
  - Bounded Trader Budget
- Which properties have SRMs following these axioms?

Essentially every continuous real-valued property over finite outcomes.

Future Work

- Relaxing our conditions
- Infinite outcomes
- Strongly convex losses
- Vector-valued properties

Summary

- Elicitable $\iff$ convex elicitable
- "Monotonize" the identification and integrate
- Essentially every continuous, real-valued property has a SRM that can be efficiently computed and allows players with arbitrarily small budget to participate in the market.

This project was funded by National Science Foundation Grant CCF-1657598.

[1] Relaxed in the paper to allow for nondifferentiable points.